

Spatiotemporally Localized Multidimensional Solitons in Self-Induced Transparency Media

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"Light bullets" are multi-dimensional solitons which are localized in both space and time. We show that such solitons exist in two- and three-dimensional self-induced-transparency media and that they are fully stable. Our approximate analytical calculation, backed and verified by direct numerical simulations, yields the multi-dimensional generalization of the one-dimensional Sine-Gordon soliton.

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The concept of multi-dimensional solitons that are localized in both space and time, alias "light bullets" (LBs), was pioneered by Silberberg [1], and has since then been investigated in various nonlinear optical media, with particular emphasis on the question of whether these solitons are stable or not. For a second-harmonic generating medium, the existence of stable two- and three-dimensional (2D and 3D) solitons was predicted as early as in 1981 [2], followed by studies of their propagation and stability against collapse [3–6], and of analogous 3D quantum solitons [7]. In a nonlinear Schrödinger model both stable and unstable LBs were found [8] and it was suggested that various models describing fluid flows yield stable 2D spatio-temporal solitons [9]. Recently, the first experimental observation of a quasi-2D "bullet" in a 3D sample was reported in Ref. [10].

In this letter we predict a new, hitherto unexplored type of LBs, obtainable by 2D or 3D *self-induced transparency* (SIT). SIT involves the solitary propagation of an electromagnetic pulse in a near-resonant medium, irrespective of the carrier-frequency detuning from resonance [11,12]. The SIT soliton in 1D near-resonant media [13] is exponentially localized and stable. In order to investigate the existence of "light bullets" in SIT, i.e. solitons that are localized in both space and time, one has to consider a 2D or 3D near-resonant medium. Here we present an approximate analytical solution of this problem, which is checked by and in very good agreement with direct numerical simulations.

Our starting point are the two-dimensional SIT equations in dimensionless form [14]

$$-i\mathcal{E}_{xx} + \mathcal{E}_z - \mathcal{P} = 0 \quad (1a)$$

$$\mathcal{P}_\tau - \mathcal{E}W = 0 \quad (1b)$$

$$W_\tau + \frac{1}{2}(\mathcal{E}^*\mathcal{P} + \mathcal{P}^*\mathcal{E}) = 0. \quad (1c)$$

Here \mathcal{E} and \mathcal{P} denote the slowly-varying amplitudes of the electric field and polarization, respectively, W is the inversion, z and x are respectively the longitudinal and transverse coordinates (in units of the effective absorption length α_{eff}), and τ the retarded time (in units of the input pulse duration τ_p). The Fresnel number F ($F > 0$), which governs the transverse diffraction in 2D and 3D propagation, is incorporated in x and the detuning $\Delta\Omega$ of the carrier frequency from the central atomic resonance frequency is absorbed in \mathcal{E} and \mathcal{P} [15]. We have neglected polarization dephasing and inversion decay, considering pulse durations that are much shorter than the corresponding relaxation times. Eqs. (1) are then compatible with the local constraint $|\mathcal{P}|^2 + W^2 = 1$, which corresponds to conservation of the Bloch vector [14].

The first nontrivial question is to find a Lagrangian representation for these 2D equations, which is necessary for adequate understanding of the dynamics. To this end, we rewrite the equations in a different form, introducing the complex variable ϕ defined as follows [16]

$$\phi \equiv \frac{1+W}{\mathcal{P}} = \frac{\mathcal{P}^*}{1-W} \iff \mathcal{P} = \frac{2\phi^*}{\phi\phi^*+1}, W = \frac{\phi\phi^*-1}{\phi\phi^*+1}. \quad (2)$$

Eqs. (1b) and (1c) can then be expressed as a single equation, $\phi_\tau + (\mathcal{E}/2)\phi^2 + (1/2)\mathcal{E}^* = 0$. Next, we define a variable f so that $\phi \equiv 2f_\tau/(\mathcal{E}f)$. In terms of f , the previous equation becomes $f_{\tau\tau} - (\mathcal{E}_\tau/\mathcal{E})f_\tau + (1/4)|\mathcal{E}|^2f = 0$. This equation is equivalent to

$$f_\tau = \frac{1}{2}\mathcal{E}g \quad (3a)$$

$$g_\tau = -\frac{1}{2}\mathcal{E}^*f, \quad (3b)$$

with $g \equiv f\phi$. Applying the same transformations to Eq. (1a) yields

$$-i\mathcal{E}_{xx} + \mathcal{E}_z - 2fg^* = 0. \quad (4)$$

The Lagrangian density corresponding to Eqs. (3) and (4) can now be found in an explicit form,

$$\mathcal{L}(x, \tau) = \frac{1}{4}\mathcal{E}_x\mathcal{E}_x^* + \frac{i}{8}(\mathcal{E}\mathcal{E}_z^* - \mathcal{E}_z\mathcal{E}^*) - \frac{i}{2}(f^*g\mathcal{E} - fg^*\mathcal{E}^*) - \frac{i}{2}(ff^* - \dot{f}\dot{f}^*) - \frac{i}{2}(gg^* - \dot{g}\dot{g}^*). \quad (5)$$

Now we proceed to search for LB solutions. Before resorting to direct simulations, we obtain an analytical approximation of the solutions. The starting point for this approximation is the well-known soliton solution for 1D SIT (the Sine-Gordon soliton) [12,14,17]

$$\mathcal{E}(\tau, z) = \pm 2\alpha \operatorname{sech}\Theta \quad (6a)$$

$$\mathcal{P}(\tau, z) = \pm 2\operatorname{sech}\Theta \tanh\Theta \quad (6b)$$

$$W(\tau, z) = \operatorname{sech}^2\Theta - \tanh^2\Theta, \quad (6c)$$

with $\Theta(\tau, z) = \alpha\tau - \frac{z}{\alpha} + \Theta_0$, and α, Θ_0 arbitrary real parameters. Equation (6a) is also called a 2π -pulse, because its area $\int_{-\infty}^{\infty} \mathcal{E}(\tau, z) d\tau = \pm 2\pi$.

Returning to the 2D SIT equations, we notice by straightforward substitution into Eqs. (3) that a 2D solution with separated variables, in the form $\mathcal{E}(\tau, z, x) = \mathcal{E}_1(\tau, z) \mathcal{E}_2(x)$ (and similarly for f and g), does not exist. To look for less obvious solutions, we first split equations (1) into their real and imaginary parts, writing $\mathcal{E} \equiv \mathcal{E}_1 + i\mathcal{E}_2$ and $\mathcal{P} \equiv \mathcal{P}_1 + i\mathcal{P}_2$:

$$\mathcal{E}_{2xx} + \mathcal{E}_{1z} - \mathcal{P}_1 = 0 \quad (7a)$$

$$\mathcal{E}_{1xx} - \mathcal{E}_{2z} + \mathcal{P}_2 = 0 \quad (7b)$$

$$\mathcal{P}_{1\tau} - \mathcal{E}_1 W = 0 \quad (7c)$$

$$\mathcal{P}_{2\tau} - \mathcal{E}_2 W = 0 \quad (7d)$$

$$W_{\tau} + \mathcal{E}_1 \mathcal{P}_1 + \mathcal{E}_2 \mathcal{P}_2 = 0. \quad (7e)$$

In the absence of the x -dependence, these equations are invariant under the transformation $(\mathcal{E}_1, \mathcal{P}_1) \leftrightarrow (\mathcal{E}_2, \mathcal{P}_2)$. This suggests a 1D solution in which real and imaginary parts of the field and polarization are equal, $\mathcal{E}_1 = \mathcal{E}_2$ and $\mathcal{P}_1 = \mathcal{P}_2$, and such that the total field and polarization reduce to the SG solution (6). Our central result is an approximate but quite accurate (see below) extension of this solution, applicable to the 2D SIT equations. In terms of the original physical variables it is given by

$$\mathcal{E}(\tau, z, x) = \pm 2\alpha \sqrt{\operatorname{sech}\Theta_1 \operatorname{sech}\Theta_2} \exp(-i\Delta\Omega\tau + i\pi/4) \quad (8a)$$

$$\begin{aligned} \mathcal{P}(\tau, z, x) = & \pm \sqrt{\operatorname{sech}\Theta_1 \operatorname{sech}\Theta_2} \{ (\tanh\Theta_1 + \tanh\Theta_2)^2 + \\ & \frac{1}{4}\alpha^2 C^4 [(\tanh\Theta_1 - \tanh\Theta_2)^2 - \\ & 2(\operatorname{sech}^2\Theta_1 + \operatorname{sech}^2\Theta_2)]^2 \}^{1/2} \exp(-i\Delta\Omega\tau + i\mu) \end{aligned} \quad (8b)$$

$$\begin{aligned} W(\tau, z, x) = & [1 - \operatorname{sech}\Theta_1 \operatorname{sech}\Theta_2 \{ (\tanh\Theta_1 + \tanh\Theta_2)^2 + \\ & \frac{1}{4}\alpha^2 C^4 [(\tanh\Theta_1 - \tanh\Theta_2)^2 - \\ & 2(\operatorname{sech}^2\Theta_1 + \operatorname{sech}^2\Theta_2)]^2 \}]^{1/2}, \end{aligned} \quad (8c)$$

with

$$\begin{aligned} \Theta_1 &= \alpha\tau - \frac{z}{\alpha} + \Theta_0 + Cx \\ \Theta_2 &= \alpha\tau - \frac{z}{\alpha} + \Theta_0 - Cx, \\ \mu &\equiv \arctan(\mathcal{P}_2/\mathcal{P}_1). \end{aligned}$$

Here α, Θ_0 and C are real constants. Equations (8) satisfy the two-dimensional SIT equations (7a) and (7b) and

obey the normalization condition $\mathcal{P}_1^2 + \mathcal{P}_2^2 + W^2 = 1$. They reduce to the Sine-Gordon solution for $C = 0$. The accuracy to which Eqs. (8) satisfy Eqs. (7c)-(7e) is $O(\alpha C^2)$, which requires that $|\alpha|C^2 \ll 1$. This is the single approximation made. Numerical simulations discussed later on verify that Eq. (8) indeed approximates the exact solution of Eq. (7) to a high accuracy. In addition, we have checked that substitution of (8) into the Lagrangian (5) and varying the resulting expression with respect to the parameters α and C yields zero. This "variational approach" is commonly used to obtain an approximate "ansatz" solution to a set of partial differential equations in Lagrangian representation [18]. Equations (8) represents a *light bullet*, which decays both in space and time and is stable for all values of z . The latter follows directly from (8a) and also from the Vakhitov-Kolokolov stability criterion [19].

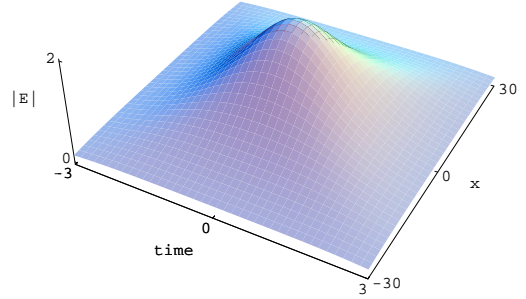


FIG. 1. The electric field in the 2D "light bullet", $|\mathcal{E}|$, as a function of time τ (in units of the input pulse duration τ_p) and transverse coordinate x (in units of the effective absorption length α_{eff}) after propagating the distance $z = 1000$. Parameters used correspond to $\alpha = 1$, $C = 0.1$ and $\Theta_0 = 1000$.

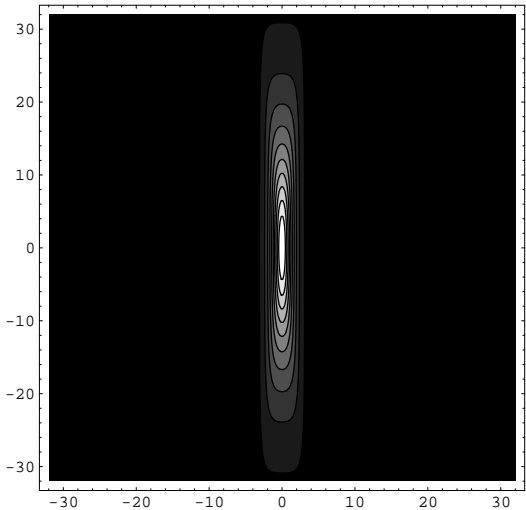


FIG. 2. Contourplot of Fig. 1 in the (τ, x) -plane. Regions with lighter shading correspond to higher values of the electric field. Note the different time scale than that of Fig. 1.

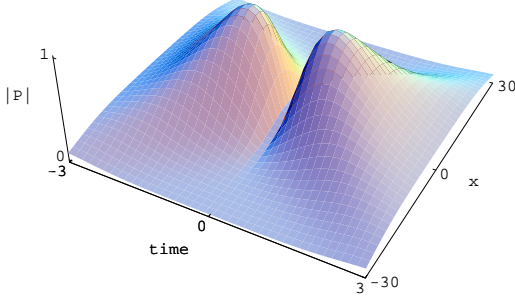


FIG. 3. The polarization in the 2D "bullet", $|\mathcal{P}|$, as a function of time τ and transverse coordinate x . Parameters used are the same as in Fig. 1.

Figs. 1-3 show the electric field and polarization, generated by direct numerical simulation of the 2D SIT equations (1) at the point $z = 1000$, using (8) as an initial ansatz for $z = 0$. To a very good accuracy (with a deviation $< 1\%$), they still coincide with the initial configuration and analytic prediction (8). The electric field has a typical shape of a 2D LB, localized in time and the transverse coordinate x , with an amplitude 2α and a nearly sech-form cross-section in a plane in which two of the three coordinates τ , z and x are constant. The ratio C/α determines how fast the field decays in the transverse direction. For $|C/\alpha| \ll 1$ (then $|C| < 1$, as $|\alpha|C^2 \ll 1$), we have a relatively rapid decay in τ and slow fall-off in the x -direction, as is seen in Fig. 1. In the opposite case, $|C/\alpha| \gg 1$, the field decays more slowly in time and faster in x . The polarization field has the shape of a double-peaked bullet. Its cross-section at constant x displays a minimum at $\Theta_{\min} \approx 0$, where $|\mathcal{P}(\Theta_{\min})| \approx 0$, and maxima at $\Theta_{\pm} = \pm \text{Arcosh}(\sqrt{2})$, where $|\mathcal{P}(\Theta_{\pm})| \approx 1$. The field and polarization decay in a similar way, which is a characteristic property of SIT [14]. Also the inversion decays both in time and in x , but to a value of -1 instead of zero, corresponding to the atoms in the ground state at infinity. A numerical calculation of the field area at $x = 0$ yields $\int_{-\infty}^{\infty} d\tau |\mathcal{E}(\tau, z, 0)| = 6.28 \pm 0.05 \approx 2\pi$, irrespective of z . By analogy with the SG soliton, one might thus name this a "2 π bullet".

We have also numerically obtained *axisymmetric* stable LBs in a 3D SIT medium, see Fig. 4. The 3D medium is described by Eqs. (1) with the first one replaced by

$$-i(\mathcal{E}_{rr} + r^{-1}\mathcal{E}_r) + \mathcal{E}_z - \mathcal{P} = 0, \quad (9)$$

where $r \equiv \sqrt{x^2 + y^2}$ is the transverse radial coordinate. Searching for an analytic 3D bullet solution in the trans-

verse plane proves to be difficult. However, in the limit of either large or small r , an approximate analytic solution may be found. For large r , it again takes the form (8), but now with $\Theta_1 = \alpha\tau - z/\alpha + \Theta_0 + Cr$ and $\Theta_2 = \alpha\tau - z/\alpha + \Theta_0 - Cr$, where α , Θ_0 , and C are constants, $|\alpha|C^2 \ll 1$, and it is implied $r \gg 1/|C|$. It is in sufficiently good agreement (deviations $< 5\%$) with results of simulation of the 3D equations, using this solution as an initial ansatz. Comparison of Figs. 1 and 4 shows that the 2D and 3D bullets have similar shapes, but the 3D one decays faster in the radial direction for small r than the 2D bullet in its transverse direction.

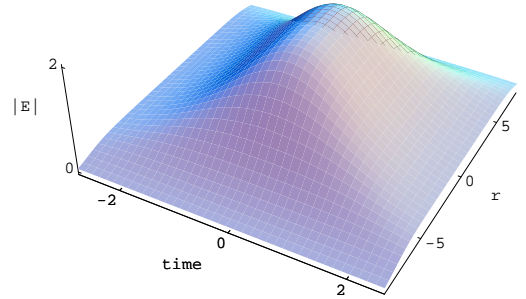


FIG. 4. The electric field in the 3D "light bullet", $|\mathcal{E}|$, as a function of time τ and transverse radial variable r after propagating the distance $z = 1000$. Parameters used correspond to $\alpha = 1$, $C = 0.1$ and $\Theta_0 = 1000$.

For constant τ , the 2D and 3D bullets are localized in both the propagation direction z and the transverse direction(s). One may also ask whether there exist SIT solitons which are traveling (plane) waves in z and localized in x (and y). Using a symmetry argument, it is straightforward to prove that they do *not* exist. Starting from the SIT equations (1) (in 2D, the 3D case can be considered analogously) we adopt a plane-wave ansatz for \mathcal{E} and \mathcal{P} , changing variables as follows: $x \rightarrow \sqrt{k}x$ (assuming $k > 0$), $\mathcal{E}(\tau, z, x) \rightarrow \mathcal{E}(\tau, x) \exp(-ikz)$, $\mathcal{P}(\tau, z, x) \rightarrow k^{-1}\mathcal{P}(\tau, x) \exp(-ikz)$, and $W(\tau, z, x) \rightarrow k^{-1}W(\tau, x)$. The equations for the real and imaginary parts of the field then become

$$\mathcal{E}_{2xx} - \mathcal{E}_2 - \mathcal{P}_1 = 0 \quad (10a)$$

$$\mathcal{E}_{1xx} - \mathcal{E}_1 + \mathcal{P}_2 = 0, \quad (10b)$$

with the equations for \mathcal{P}_τ and W_τ given by (7c)-(7e). Using the transformation $(\mathcal{E}_1, \mathcal{P}_1) \leftrightarrow (\mathcal{E}_2, \mathcal{P}_2)$, which leaves the last three equations invariant but changes the first two, one immediately finds that (10) only admits the trivial solution $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{P}_1 = \mathcal{P}_2 = 0$, $W = -1$.

The observation of "light bullets" in a SIT process requires high input power of the incident pulse and high

density of the two-level atoms in the medium, in order to achieve pulse durations short compared to decoherence and loss times. These requirements are met e.g. for alkali gas media, with typical atomic densities of $\sim 10^{11}$ atoms/cm³ and relaxation times ~ 50 ns [20], and for optical pulses generated by a laser with pulse duration $\tau_p < 0.1$ ns. In order to include transverse diffraction, the incident pulse should be of uniform transverse intensity and satisfy $\alpha_{\text{eff}} d^2 / \lambda < 1$ [20], where λ and d are its carrier wavelength and diameter respectively [20]. The parameter α in the solution (8), which determines the amplitude of the bullet and its decay in time, corresponds to $\alpha \sim \kappa_z \tau_p v_p$ [13], with κ_z the wavevector component along the propagation direction z and v_p the velocity of the pulse in the medium, and can thus be controlled by the incident pulse duration and velocity. The parameter $C \sim \kappa_x L_x$, where κ_x is the transverse component of the wavevector and L_x is the spatial transverse width of the pulse, is also controlled by the characteristics of the incident pulse and should satisfy the condition $\kappa_z \kappa_x^2 L_z L_x^2 \ll 1$. For a homogeneous (atomic beam) absorber, the effective absorption length $\alpha_{\text{eff}} \sim 10^4$ m⁻¹ and the Fresnel number F can range from 1 to 100 [20]. The bullets then decay on a time scale of $t \sim 1 - 10 \tau_p \sim 10$ ns and transverse length of $x \sim 0.1 - 1$ mm, which is well within experimental reach.

In conclusion, we predict the existence of fully stable "light bullets" in 2D and 3D self-induced transparency media. The prediction is based on an approximate analytical solution of the multi-dimensional SIT equations and verified by direct numerical simulation of these PDE's. Our results suggest an experiment aimed at detection of this "bullet" in an SIT-medium and opens the road for analogous searches for "light bullets" in other nonlinear optical processes, such as, e.g., stimulated Raman scattering, which is analogous to SIT.

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